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Fractional D3-branes in diverse backgrounds

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Abstract: In the first part of this article, we find fractional D3-branes supergravity solutions on orientifolded $\mathbb{C}^2/\mathbb{Z}_2$ orbifolds of type-IIB string theory. The one-loop corrected gauge couplings for the symplectic or orthogonal groups living on the D-branes are reproduced on the world-volume of probes. In the second part of the paper, we construct a D3-brane solution on the two-centers Taub-NUT manifold which interpolates between fractional D3-branes in the ALE space limit and a T-dual smeared type-IIA configuration. Then, we lift this configuration to M-theory and comment on the connections with wrapped M5-branes solutions.

Keywords: D-branes, String Duality, AdS-CFT and dS-CFT Correspondence.

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1. Introduction

During the last couple of years, the study of gauge/gravity duality has partly shifted from the original AdS/CFT correspondence [1] and its less supersymmetric cousins to more realistic, non-conformal cases. Different approaches has been proposed and explored, with configurations preserving $\mathcal{N} = 2$, $\mathcal{N} = 1$ or no supersymmetries: mass deformations of a conformal field theory [2, 3, 4], NS5-branes or M5-branes wrapped on a Riemann surface [5, 6, 7], fractional D3-branes on orbifold spaces or conifolds [8]–[19] to cite several of those constructions. Typically, the "naive" supergravity solution displays naked singularities. However, the singularities are usually the manifestation of interesting and non-trivial phenomena in the infrared of the dual gauge theory and their resolutions in supergravity should take into account such effects. A typical example is the $\mathcal{N} = 1$ solution of [10], where the chiral symmetry breaking of the field theory in the IR removes the singularity by deforming the conifold.

This example uses fractional D3-branes which first appeared the context of string theory defined on orbifold spaces [20]. On a conifold, one does not have anymore a microscopic description of the fractional D-branes and must rely on the pure supergravity solutions. One the other hand, for fractional D3-branes in type-IIB theory on the orbifold space $\mathbb{C}^2/\mathbb{Z}_N$, one can use the conformal field theory technology. Indeed, these D-branes have an exact description in string theory as boundary states [21]. Despite being less interesting for real world applications than their $\mathcal{N} = 1$ counterparts, these D3-branes configurations, whose world-volume gauge theory possesses an $\mathcal{N} = 2$ supersymmetry, could be seen as tools to study and to test the gauge/gravity correspondence away from its conformal limit, like the original, maximally (super-)symmetric D3-branes were useful for the conformal case. Indeed, the larger amount of preserved supersymmetries gives a better control of the theories on both sides; in the gauge theory, we still have non-renormalization theorems, telling us, for instance, that the gauge couplings only receive perturbative corrections at one-loop. Moreover, its non-perturbative corrections could be, in principle, obtained using the Seiberg-Witten curve of the gauge theory [22]. On the gravity side, one could also expect that the α' corrections are better controlled, as it is well known for solutions of heterotic string theories. Moreover, these $\mathcal{N} = 2$ configurations display on their own interesting phenomena like the enhançon mechanism [23] which was argued to solve the singular behaviour of their metrics; this characteristic is a common feature of several kinds of solutions preserving $\mathcal{N} = 2$ supersymmetries presented in the literature, namely, mass deformation of an $\mathcal{N} = 4$ gauge theory [24, 25], NS5-branes wrapped on a Riemann surface [7] or fractional D3-branes [12, 15, 16]. Actually, the enhançon mechanism is the gravity manifestation of the existence of Landau poles in the gauge theory, when the one-loop correction cancels the tree-level contribution. However, we know that the complete metric on the moduli space of a $\mathcal{N} = 2$ gauge theory is not singular when Yang-Mills instantons corrections are included. It is still a challenge to describe precisely these effects from the dual point-of-view.

Among $\mathcal{N} = 2$ non-conformal configurations, the fractional D3-branes one seems to be the simplest. For instance, it is obvious to describe an arbitrary point on the Coulomb branch of the gauge theory since the *vev* of the adjoint scalars which parameterize the moduli space are proportional to the positions of the fractional D3-branes and are chosen arbitrarily to construct the solution. One the other hand, in mass deformed or wrapped NS5-branes configurations, the precise identification of the point on the Coulomb branch requires some work and, so far, it is not clear that any point can be obtained by deforming the known solutions.[1] Moreover, we can also see a fractional D-brane configuration as a "mass deformed" conformal theory: one can always start with a stack of bulk D-branes and give masses to bifundamental hypermultiplets by pulling away some fractional D-branes. At large distance, the theory will be conformal but, in the IR of the gauge theory, the D-branes sources break the conformal symmetry.

The aim of this article, which is made of two independent parts, is two-folds. In the first part, we will generalize fractional D3-branes solutions to models with orientifolds. The presence of O-planes leads to symplectic or orthogonal gauge groups on the world-volume of the D-branes. After a short review of some basic facts on orientifold projections and existence of fractional D-branes, we describe the supergravity solutions for two possible cases. Such solutions are similar to their parents type-IIB and display common features, namely singularities and enhançon; as expected, we show that the one-loop corrected gauge couplings for orthogonal and symplectic gauge groups are reproduced on the world-volume of fractional D3-branes probing these backgrounds. This result should be interpreted as a manifestation of the closed/open string duality. We also speculate on the relation between periodic monopoles and a refined solution close to the enhançon radius.

[1] An example of such deformation have been given in [7], where the *vev* are linearly distributed.

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The second part of this paper is a first step toward a comparison of the alternative approaches to describe pure SU(N) $\mathcal{N}=2$ gauge theory using supergravity. One way to study the possible connection between very different branes configurations is to uplift them to eleven dimensions where they should be described as M5-branes wrapped on manifolds in such a way that, in the IR, the world-volume theory flows to the four-dimensional gauge theory. Moreover, the M-theory point-of-view could be useful to understand the approximations made and the domains of validity of the different solutions. In particular, one can think about non-perturbative effects that are inaccessible in the type-II description. Indeed, we know that such an eleven dimensional approach has been fruitful in the past to recover non-perturbative - D-instantons - corrections to string effective actions whose form was conjectured using symmetry - self-duality - arguments [26, 27, 28]. In this article, we study the case of the fractional D3-branes, first by constructing a solution on a two-centers Taub-NUT space which interpolates between the type-IIB fractional D3-branes solution in the ALE limit and a partially smeared type-IIA configuration in another limit. Then, we lift the solution to M-theory, where it describes an M5-brane wrapped on a Riemann surface. Finally, we will comment on the relation with M5-branes solutions proposed by [29] to be dual to the pure SU(N) $\mathcal{N}=2$ gauge theory but which has received much less attention so far than fractional D3-branes and wrapped NS5-branes.

Note added in proof: while this article was reaching its final stage, appeared the preprint [58], which develops ideas similar to the ones considered in the second part by comparing fractional D-branes to wrapped branes solutions in the context of three dimensional gauge theories.

2. Fractional D-brane solutions with orthogonal and symplectic gauge groups

2.1 Orientifold projections and fractional D3-branes

We start with type-IIB string theory on the non-compact orbifold space $\mathbb{C}^2/\mathbb{Z}_2$; the directions parallel to the orbifold will be called x^6, x^7, x^8, x^9. The twisted fields of this theory, namely the NS-NS scalar b, the three blow-up modes, $\xi^{\pm}, \xi^3$, the R-R scalar c and the R-R 2-form $A^{(2)}$ fill a six dimensional $\mathcal{N}_6=(2,0)$ tensor multiplet. In order to discuss the orientifold projection, it is convenient to use the $\mathcal{N}_6=(1,0)$ supersymmetry language where this tensor multiplet is split into a tensor multiplet and a hypermultiplet. The fields b and $A^{(2)}$ make up the first while the four other scalars fill the hypermultiplet.

"Standard" orientifold projection [30, 31] keeps the hypermultiplet and projected out the tensor multiplet. Therefore, the fractional D-branes which appear is this model always come in pairs. One can see this easily using a T-dual type-IIA description. T-dualizing the orbifold space $\mathbb{C}^2/\mathbb{Z}_2$ along its U(1) direction maps the A_1 singularity to a pair of NS5-branes separated along the T-dualized direction, x^6 [32, 33]. The orientifold 9-plane splits into two orientifolds 8-planes located at antipodal points on the circle parameterized by x^6. For the standard choice, each of the NS5-branes coincides with one $O8$-plane. Now, we remind the reader that, under T-duality, the twisted fields are supposed to be mapped to differences between the fields which live on each NS5-branes and which also fill $\mathcal{N}_6 = (2,0)$ tensor multiplets [34]. For instance, the b field is related to the distance between the two NS5-branes along x^6 while the blow-up modes are mapped to the relative positions of these branes along x^7, x^8 and x^9. Therefore, for symmetry reasons, we see that, for such orientifold choice, the field b and $A^{(2)}$ are projected out.

One the other hand, the above T-dual picture suggests another choice for the orientifold projection [35, 36, 37] which will give fractional D-branes similar to their type-IIB counterparts [38, 39]. Indeed, it is possible to move the NS5-branes off the $O8$-planes in the direction x^6 and the symmetry of the problem imposes that the NS5-branes can not be moved anymore independently in the x^7, x^8 or x^9 directions. T-dualizing back to type-IIB string theory, this means that the orientifold operation now projects out the hypermultiplet of the twisted sector. The orbifold space lacks its blow-up modes but has a twisted NS-NS b field and its 2-form companion. In the following, we will always consider this second possibility and construct the fractional D-branes solutions on such backgrounds.

To get fractional D3-branes, we perform two T-dualities along two directions transverse to the orbifold; the R-R 2-form becomes a twisted 4-form under which the fractional D3-branes will be charged and which can also be Hodge-dualized to a scalar, called c below. The theory will contain $O7$-planes, and depending on the orientifold projection, D7-branes to cancel the tadpole of the R-R 8-form. More precisely, we will consider in this paper two different orientifold projections which give rise to symplectic or/and orthogonal groups on the fractional D3-branes.

The first one has an orientifold projection defined by $P_1 \equiv (1+\alpha)(1+\Omega')/4$ where α is the generator of the $\mathbb{Z}_2$ orbifold group acting on the coordinates as $x^6, x^7, x^8, x^9 \to -(x^6, x^7, x^8, x^9)$ and $\Omega' = \Omega R_{45}(-1)^{F_L}$ where Ω is the standard orientifold operator, R_{45} is the reflection in x^4, x^5 directions and $(-1)^{F_L}$ acts as -1 on the left-moving Ramond sector. The presence of these last two operators is due to the T-duality transformations. This model contains four $O7^-$-planes whose total negative charge under $C^{(8)}$ is neutralized by adding 32 D7-branes. In this background, we can have fractional D3-branes and the gauge theory which lives on the world-volume of N of those has a gauge group $\mathrm{Sp}(2N)$, with a hypermultiplet in the fundamental representation, coming from strings stretched between the D7-branes and these D3-branes. More precisely, as in type-IIB string theory, we have two different kinds of fractional D3-branes, whose world-volume theories are obtained by a projection of the gauge theories living on their type-IIB parents. For the orientifold projection P_1 both types of fractional D-branes have the same massless field content. Therefore, for $2N_1$ fractional D3-branes of the first kind and $2N_2$ of the second kind (we are counting also their images under Ω'), the gauge theory will have a group $\mathrm{Sp}(2N_1) \times \mathrm{Sp}(2N_2)$ with one hypermultiplet in the bifundamental, two in the $\square_1$ and two in the $\square_2$. The minimally covariantized action of the linear couplings of these D3-branes to the background fields can be extracted from their boundary state description:

$$\mathcal{S}_{1/2} = -\frac{T_3}{\sqrt{2}\kappa_{\mathrm{orb}}}\left(\int d^4\sigma\, e^{-\phi}\sqrt{-\det(g+F)}\left(1 \pm \frac{\tilde{b}}{2\pi^2\alpha'}\right) - \int C^{(4)}\left(1 \pm \frac{\tilde{b}}{2\pi^2\alpha'}\right) \pm \frac{A^{(4)}}{2\pi^2\alpha'}\right) \tag{2.1}$$

where g is the induced metric on the world-volume of the D-brane: $g_{\alpha\beta} \equiv G_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu$.

The second orientifold projection we will consider does not contain any D7-branes. The operator is defined by $P_2 \equiv (1+\alpha)(1+\beta\Omega')$ where $\beta^2 = \alpha$ and does not induce any R-R tadpole. In the T-dual picture, the corresponding orientifold plane splits into two $O6$-planes, one negatively charged and the other oppositely charged under the R-R 7-form. Again, in this background, we have two kinds of fractional D3-branes; however, the orientifold projection now gives rise to a $\mathrm{Sp}(2N)$ gauge theory on the first type and to a $\mathrm{SO}(2N)$ gauge group on the other. The T-dual description gives again a simple geometrical interpretation of this result: the first case corresponds to D4-branes intersecting the $O6^-$-plane as for P_1, while the second comes from a D4-brane intersecting the positively charged O-plane. The most general configuration will have N_1 fractional D3-branes of the first kind and N_2 of the second type (plus their images under Ω'), leading to an $\mathrm{Sp}(2N_1) \times \mathrm{SO}(2N_2)$ gauge theory with one hypermultiplet in the bifundamental. The superconformal theories constructed using D3-branes in these models have been studied in [40, 41]. In the context of the AdS/CFT correspondence, supergravity solutions for these conformal configurations have been given in [42].

2.2 Supergravity solutions and one-loop metric on the moduli space

2.2.1 The field equations

Since we know their type-IIB counterparts, the supergravity descriptions of these fractional D3-branes are easy to find. Indeed, we have just seen that they couple to the same supergravity fields. The fields projected out by the orientifold operators, namely, the untwisted NS-NS 2-form, the untwisted R-R 2-form, the twisted R-R 2-form and the blow-up modes were already inactive in the type-IIB case. Therefore, the homogeneous field equations will be identical. These equations for the scalar fields can be read in [17, 18]

$$d^{\star} d\tau = 0\,, \tag{2.2}$$

$$d^{\star 6} d\gamma + iF_{(5)} \wedge (dc + \tau db) - b\, d^{\star 6} d\tau = 0\,, \tag{2.3}$$

where we have introduced the complex combinations

$$\tau = C^{(0)} + ie^{-\phi} \qquad \text{and} \qquad \gamma = c + \tau b$$

and defined the self-dual 5-form

$$F^{(5)} = dC^{(4)} + {}^{\star}dC^{(4)}\,.$$

Requiring that a fraction of the supersymmetries is preserved by the solution implies that the scalar fields τ and γ must be holomorphic functions of z. This constraint has been used to derive the equations (2.2) and (2.3). The two-dimensional homogeneous Laplace equation (2.2) is automatically satisfied and the precise form of τ must come from other arguments, one being that it has to reflect the presence of sources for the untwisted R-R scalar. An other requirement is that its imaginary part, $e^{-\phi}$, must stay positive. Finally, one can also use its modular properties under the group $\mathrm{SL}(2,\mathbb{Z})$.

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Using the standard metric and self-dual 5-form ansatz for D3-branes on top of D7-branes/O7-plane [43],

$$\begin{aligned} ds^2 &= H^{-1/2}\eta_{\alpha\beta}dx^\alpha dx^\beta + H^{1/2}(e^\psi dz d\bar{z} + \delta_{ij}dx^i dx^j)\,, \\ F^{(5)} &= d(H^{-1}dx^0 \wedge \ldots \wedge dx^3) + \,^\star d(H^{-1}dx^0 \wedge \ldots \wedge dx^3)\,, \end{aligned} \tag{2.5}$$

in the Einstein and 5-form equations leads[2]

$$dx^4 \wedge dx^5 \left(e^\psi \delta^{ij}\partial_i\partial_j + \partial_z\partial_{\bar{z}}\right) H - \delta(x^6)\ldots\delta(x^9)db\wedge dc = 0 \tag{2.6}$$

for the harmonic function $H(z,\bar{z},x^i)$. Finally, ψ is a function of z and $\bar{z}$ which obeys the equation

$$\partial_z\partial_{\bar{z}}\left(\psi - \log\tau_2\right) = 0\,. \tag{2.7}$$

Compared to the fractional D3-branes on top of D7-branes studied in [17, 18], the difference lies in the boundary conditions (or, equivalently, source terms in the right hand side of the above field equations) and in the geometrical symmetries imposed by the orientifold projections.

2.2.2 $\mathrm{Sp}(2N_1) \times \mathrm{Sp}(2N_2)$ gauge theory

Besides the usual orbifold projection $(1+\alpha)$, the orientifold projection P_1 has a geometrical action on the coordinates $z = x^4 + ix^5 \to -z$, due to the presence of the reflection operator R_{45} in Ω'. One of the four orientifold 7-plane is located at the fixed point $z = 0$ of this operator, the three other being rejected at infinity in the decompactified limit of the theory. In this study, we will forget them and consider only the O7-plane at the origin and its eight D7-branes companions. If one only imposes the global cancellation of the untwisted R-R tadpole, these D7-branes can be located away from the origin of the z-plane. However, the R_{45} symmetry imposes a constraint on their positions. Moreover, these D7-branes carry a charge under the twisted R-R field which, in absolute value, is one quarter of the charge of a fractional D3-brane. Before doing the T-dualities along x^4 and x^5, twisted R-R tadpole cancellation was a requirement for the consistency of the theory and imposed that one half of the D9-branes carries a positive charge and the other half a negative charge.

In the decompactified theory obtained after T-duality, this choice is no longer required. However, in this article, we will still assume that the twisted charge induced by the D7-branes vanishes locally: more precisely, we will choose to pair the D-branes in such way that a D-brane with positive twisted charge is at the same location as a D-brane with negative twisted charge. The other constraints on the location of the D7-branes are due to the orientifold and orbifold R_{45} symmetries. The orientifold symmetry tells us that only four D-branes (we are working on the covering space) are real and could have any position. Requiring the R_{45} symmetry implies that if a D7-brane is at a position $z \neq 0$, it must have an image at $-z$. Finally, these three constraints reduces the modules corresponding to the D7-branes positions to only one, that we will called $z = m$, all the other positions being related to this one by the symmetries and local cancellation of the twisted charge requirements.

[2] In this article, we will only write the homogeneous equations. The presence of the D-brane sources for the fields will be imposed at the level of the boundary conditions.

Separating the D7-branes from the O7-plane generates non trivial dilaton and untwisted R-R scalar. The presence of these sources for the R-R field leads to the following asymptotic behaviour:

$$\tau(z) \sim i\left(1+\frac{g_s^B}{2\pi}\Big(8\log(z)-4\log(z^2-m^2)\Big)\right) \tag{2.8}$$

which displays the expected monodromy. We have also taken into account a constant background value for τ_2. However, the function (2.8) does not always have a positive imaginary part, and, therefore can not be the complete answer.

This problem has been solved in [44], where, using its modular properties, a formula has been proposed for τ:

$$j\left(\frac{\tau(z)}{g_s^B}\right)=e^{\frac{2\pi}{g_s^B}}z^8\left(z^2-m^2\right)^{-4} \tag{2.9}$$

j is the modular invariant function and τ is restricted to the fundamental domain $\mathcal{F}$ of the modular group. Indeed, j induces a one-to-one mapping between $\mathcal{F}$ and the z-plane. At large τ_2, one obtains the result (2.8) but, even if this approximation will be adequate for the purpose of this article, one should keep in mind that it is not the whole solution and that corrections could be needed to tackle other problems.

The only sources for the twisted field equation (2.3) are the actions (2.1) of the two kinds of fractional D3-branes. The positions of the D-branes of the first kind will be denoted $\pm z_{1p}$ for $p=1,\ldots,N_1$ while, for the other set, we will use $\pm z_{2q}$ for $q=1,\ldots,N_2$. As in the previous case, this information leads to the formula:

$$\gamma(z) \sim 4i\pi g_s^B\alpha' \sum_{\epsilon=\pm1}\left(\sum_{p=1}^{N_1}\log\frac{(z+\epsilon z_{1p})}{\Lambda}-\sum_{q=1}^{N_2}\log\frac{(z+\epsilon z_{2q})}{\Lambda}\right) \tag{2.10}$$

from which we can extract the twisted scalar b and the twisted 4-form $A^{(4)}$ after an Hodge duality:

$$\begin{aligned}
\tilde{b} &\sim 4\pi g_s^B\alpha' \,\mathrm{Re}\sum_{\epsilon=\pm1}\left(\sum_{p=1}^{N_1}\log\frac{(z+\epsilon z_{1p})}{\Lambda}-\sum_{q=1}^{N_2}\log\frac{(z+\epsilon z_{2q})}{\Lambda}\right)\,,\\
A^{(4)} &\sim 4\pi g_s^B\alpha' \,\mathrm{Re}\sum_{\epsilon=\pm1}\left(\sum_{p=1}^{N_1}\log\frac{(z+\epsilon z_{1p})}{\Lambda}-\sum_{q=1}^{N_2}\log\frac{(z+\epsilon z_{2q})}{\Lambda}\right)dx^0\wedge\ldots\wedge dx^3\,.
\end{aligned}$$

As explained in the appendix of [45], this profile only encodes one-loop open string effects. However, usually [12, 13], one takes this formula as the full solution to the field equation. This choice can be justified if one insists on preserving the rotational symmetry in the z-plane when all the D-branes are located at $z=0$. Indeed, other holomorphic solutions involving subleading corrections to (2.10) breaks this symmetry, which, from the gauge theory point-of-view, corresponds to the U(1) of the R-symmetry group. As we will see below, this solution contains enough information about the gauge coupling renormalization as predicted by open string/closed string duality and, from the gauge theory point-of-view, we do not expect any higher order perturbative corrections to it. However, this argument does not rule out the existence of exponentially vanishing corrections of order $(\Lambda/z)^n$, $n > 0$, which could be attributed to D-instanton effects. Finally, in the light of the expected gauge/gravity correspondance, one should ask how these D-instanton corrections could be related to non-perturbative effects on the Yang-Mills side. We will not address this question in this article, saying only that, in this approach, the mapping to Yang-Mills instantons is far from being obvious, as recently discussed in the literature for the $\mathcal{N} = 1$ cases [46].

Actually, in the context of fractional D2-branes [47],[3] it has been argued recently that the leading order solution receives corrections which are exponentially small at infinity but become important at some finite distance from the core. The idea was to "improve" the effective description by going to a special point of the moduli space of the type-IIA on $T^4/\mathbb{Z}_2$ string theory where fractional D0-branes become massless. Indeed, the breakdown of the supergravity description in the case of fractional D2-branes corresponds to the appearance of new massless states at a special radius r_e, due to fractional D0-branes, which had been integrated out in the standard supergravity action. At this radius, a U(1) gauge theory (corresponding to the R-R 1-form $A^{(1)}$ under which the fractional D2-branes are magnetically charged) gets enhanced to a SU(2) group, the new massless states playing the role of W-bosons. A correct way to deal with this problem is to go directly to the SU(2) point of the moduli space of the string theory and search for a solution of the new field equations. In this case, the twisted fields, b and $A^{(1)}$, obey a SU(2) Bogomolnyi equation. Its analytic solution is known for a magnetic charge equals to one. Asymptotically, the solution looks like the Dirac monopole but, close to the enhançon radius, non-perturbative effects due to the fractional D0-branes completely change the picture and lead to a smooth, non-singular metric (except obviously at the origin, where we always have a singularity, as for any D-brane). The twisted NS-NS field b plays the role of a Higgs field and, away from the special radius r_e, the SU(2) group is broken to a U(1) by the Higgs mechanism and the fractional D0-branes get a mass proportional to b.

In the case of fractional D3-branes, we expect that similar effects will also solve the singularities of the metric, by taking into account the fact that fractional D-instantons become massless at $r = r_e$. More precisely, one expects the modified solution for the twisted fields to be given by the periodic monopole solutions considered by [49], in the limit of a vanishing circle: in this paper, the authors discuss solutions to the non-abelian Bogomolnyi equation on $\mathbb{R}^2 \times S^1$. In the language of the last paragraph, this would describe fractional D2-branes with a transverse direction (also transverse to the orbifold) compactified. In the zero radius limit, we obtain the T-dual configuration, namely fractional D3-branes. Obviously the asymptotics, given by the Dirac solution, coincide with (2.10) but, unfortunately, the full solution is not known, even in the case of a charge one periodic monopole. However, one expects that the solution will be modified at short distances, like for the t'Hooft-Polyakov

[3] In the paper [47], it was believed that the Harvey-Liu solution [48] was mapped under heterotic on T^4/type-IIA on $K3$ duality to the D6-brane wrapped on $K3$ configuration of [23] but, actually, it corresponds to the fractional D2-brane case. The wrapped D6-brane configuration is mapped to a superposition of H and KK-monopoles in the heterotic string and its "non-abelian" generalization is not known so far.

monopole, in such a way that the metric singularities are smooth out. More speculatively, one could conjecture that the exact solution is such that the cascading-like behaviour of the D3-brane charge discussed in [13] and refuted in [16], completely disappears, since we do not expect such a Seiberg duality in a $\mathcal{N} = 2$ gauge theory. It would be interesting to investigate these questions further but, for the purpose of this article, we stop these speculations now and consider only the formula (2.8) and (2.10).

Using a D-brane probe, we will show that these asymptotics carry enough information to reproduce the one-loop corrected effective action on the moduli space of the $\mathrm{Sp}(2N_1)$ gauge theory with $2N_2+2$ hypermultiplets in the fundamental representation. The Laplace equation (2.6) for the harmonic function H can not be solved explicitly in the case of D7-branes away from the origin of the z-plane but will not be needed. When the untwisted R-R charge is zero, one can choose ψ=0 and the harmonic function will be similar to the case of type-IIB on $\mathbb{C}^2/\mathbb{Z}_2$ [12].

To probe the background, we consider a fractional D3-brane of the first kind located at a position z. Inserting into its action $\mathcal{S}_1$ the supergravity fields, we can extract the gauge coupling and the kinetic term for the adjoint scalar fields Φ^a which belong to the vector multiplet. The expansion of the action to quadratic order in the gauge and scalar fields leads to:

$$\begin{aligned}\mathcal{S} &= -\frac{1}{8\pi g_s^B}\int d^4x e^{-\phi}\left(1+\frac{\tilde{b}}{2\pi^2\alpha'}\right)\left(\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}+\frac{1}{2}\partial_\alpha\Phi^a\partial^\alpha\Phi^a\right)\\ &= -\frac{1}{g_{\mathrm{YM}}^2(z)}\int d^4x\left(\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}+\frac{1}{2}\partial_\alpha\Phi^a\partial^\alpha\Phi^a\right)\end{aligned} \tag{2.11}$$

with

$$\begin{aligned}\frac{1}{g_{\mathrm{YM}}^2(z)} &= \frac{1}{8\pi g_s^B}+\frac{1}{8\pi^2}\mathrm{Re}\sum_{\epsilon=\pm1}\left(\sum_{p=1}^{N_1}2\log\frac{(z+\epsilon z_{1p})}{\Lambda}-\sum_{q=1}^{N_2}2\log\frac{(z+\epsilon z_{2q})}{\Lambda}\right)+\\ &\quad +4\log\left(\frac{z}{\Lambda}\right)-\sum_{i=1,2}\log\frac{(z^2-m_i^2)}{\Lambda^2}\end{aligned} \tag{2.12}$$

This formula is the one-loop corrected gauge coupling of the symplectic gauge theory on its Coulomb branch [50], where the *vev* of its adjoint scalars are proportional to z_{1p}, with $2N_2+2$ fundamental hypermultiplets. This one-loop renormalized gauge coupling displays Landau poles where the enhançon mechanism has been argued to take place. However, we know from the gauge theory point-of-view that the complete moduli space metric, which includes non-perturbative - Yang-Mills instantons - corrections is a smooth manifold, something that our solution is not able to capture.

2.2.3 $\mathrm{Sp}(2N_1)\times\mathrm{SO}(2N_2)$ gauge theory

The set of fixed points of the other orientifold projection, P_2, reduces to the plane left $x^6 = x^7 = x^8 = x^9 = 0$ left invariant by the orbifold operator α. There is no dilaton tadpole and the complex scalar τ is a constant. However, the orientifold projection induces a background twisted charge [35, 36, 40], which is precisely twice the charge of a fractional D3-brane of the first kind. One of the consequences of this background charge is that the conformal field theory case requires fractional D-branes and not just bulk D-branes: it is obtained for $N_2 = N_1 + 1$. A solution to the twisted field equation which correctly encodes the presence of these sources reads

$$\gamma(z) = 4i\pi g_s^B \alpha' \left(\sum_{\epsilon=\pm 1} \left(\sum_{p=1}^{N_1} \log \frac{(z+\epsilon z_{1p})}{\Lambda} - \sum_{q=1}^{N_2} \log \frac{(z+\epsilon z_{2q})}{\Lambda} \right) + 2\log \frac{z}{\Lambda} \right). \tag{2.13}$$

Again, we can probe this supergravity background using the two different kinds of fractional D3-branes. Extracting the kinetic terms for scalar fields on the first kind of probe located at z leads to

$$\begin{aligned} \mathcal{S}_1 = & -\left(\frac{1}{8\pi g_s^B} + \frac{1}{8\pi^2} \mathrm{Re} \sum_{\epsilon=\pm 1} \left(\sum_{p=1}^{N_1} 2\log \frac{(z+\epsilon z_{1p})}{\Lambda} - \sum_{q=1}^{N_2} 2\log \frac{(z+\epsilon z_{2q})}{\Lambda} \right) + 4\log \frac{z}{\Lambda} \right) \times \\ & \times \int d^4\sigma \left(\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} + \frac{1}{2} \partial_\alpha \Phi^a \partial^\alpha \Phi^a \right) \end{aligned} \tag{2.14}$$

while, for the other probe,

$$\begin{aligned} \mathcal{S}_2 = & -\left(\frac{1}{8\pi g_s^B} + \frac{1}{8\pi^2} \mathrm{Re} \sum_{\epsilon=\pm 1} \left(\sum_{q=1}^{N_2} 2\log \frac{(z+\epsilon z_{2q})}{\Lambda} - \sum_{p=1}^{N_1} 2\log \frac{(z+\epsilon z_{1p})}{\Lambda} \right) - 4\log \frac{z}{\Lambda} \right) \times \\ & \times \int d^4\sigma \left(\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} + \frac{1}{2} \partial_\alpha \Phi^a \partial^\alpha \Phi^a \right) \end{aligned} \tag{2.15}$$

In these expressions, we recognize respectively the one-loop corrected metric on the moduli space of the $(\mathrm{Sp}(2N_1) + 2N_2 \,\square_1)$ and $(\mathrm{SO}(2N_2) + 2N_1 \,\square_2)$ gauge theories.

To conclude this first part, we must mention that there is (at least) another way to obtain fractional D3-branes with orthogonal or symplectic gauge groups. The construction, which has been described in [51] for the case of type-IIB string theory on $\mathbb{C}^2/\mathbb{Z}_{2N}$, uses an $O3$-plane. For $N = 1$, the gauge theory which lives on the two possible kinds of fractional D-branes is $\mathrm{Sp}(2N_1) \times \mathrm{SO}(2N_2)$ with a hypermultiplet in the bifundamental. In this paper, one can also see that the orientifold projection, called $\Omega\Pi$, does not introduce any untwisted tadpoles but gives the same background twisted charge in the sector $k = N$ as the P_2 projection we considered in this section. Therefore, the supergravity solution and the gauge couplings renormalizations will be identical.

3. Fractional D-branes versus wrapped M5-branes

3.1 D3-branes on a two-centers Taub-NUT space

Alternative approaches to describe non-conformal pure $\mathcal{N} = 2$ SU(N) super-Yang-Mills theories using supergravity solutions have been discussed recently [4, 6, 7, 29]. It would be interesting to make connections between these various models and to completely understand which points on the moduli space of the gauge theory they describe. The aim of this section is to provide a first step in this direction by comparing the fractional D3-brane supergravity solution of type-IIB on the non-compact orbifold $\mathbb{C}^2/\mathbb{Z}_2$ to another proposal, namely the M5-brane wrapped on a Riemann surface of [29].

The strategy we will adopt is to lift the type-IIB solution to eleven dimensions. To do this, we will first replace the orbifold singularity by the two-centers Taub-NUT space. Indeed, we know that, on one hand, a k-centers Taub-NUT manifold degenerates into a $\mathbb{C}^2/\mathbb{Z}_k$ orbifold in a specific limit and, on the other hand, it is T-dual to k NS5-branes delocalized along the T-dualized direction [32, 33]. Then, we construct fractional D3-brane solutions on this ALF space, using the method described in [18] to obtain the supergravity solution for D3-branes with fluxes on the Eguchi-Hanson manifold. Performing a T-duality along the fibered U(1) of the Taub-NUT manifold, we will obtain a type-IIA solution describing D4-branes stretched between NS5-branes. Finally, using the type-IIA/M-theory duality dictionary, we will lift the solution to eleven dimensions.

Let us first begin by reviewing some properties of the two-centers Taub-NUT space [52], which will be useful to find the D3-brane solution. The metric of the Taub-NUT manifold parameterized by the coordinates $(x^6, \ldots, x^9)$ and with centers located at $x^i = x^i_1$, x^i_2 for $i = 7, 8, 9$, can be written as:

$$ds^2_{\text{Taub-NUT}} = V\delta_{ij}dx^i dx^j + V^{-1}(dx^6 + A_i dx^i)^2 \tag{3.1}$$

with

$$V = 1 + \sum_{a=1,2} \frac{R}{|x^i - x^i_a|} \qquad \text{and} \qquad \partial_i V = \epsilon_{ijk}\partial_j A_k\,. \tag{3.2}$$

In order to avoid conical singularities at $r = 0$, the coordinate x^6 must have periodicity $4\pi R$. The metric is asymptotically flat and approaches the cylinder $\mathbb{R}^3 \times S^1_{4\pi R}$ at large $|x|$. When the centers become coincident, this metric has a $\mathbb{Z}_2$ singularity at $r = 0$. In this case, the orbifold limit is obtained by sending the radius R to infinity. More precisely, if one defines the new coordinates (r, θ, ϕ, ψ)

$$x^6 = 2R\psi\,, \quad x^7 = r\cos\theta\,, \quad x^8 = r\sin\theta\,, \quad \cos\phi \quad \text{and} \quad x^9 = r\sin\theta\sin\phi$$

in the limit $R \to \infty$, the metric (3.1) with $x_1 = x_2 = 0$ degenerates to

$$ds^2 \simeq 2R\left(\frac{1}{r}\left(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right) + r(d\psi + \cos\theta d\phi)^2\right) \tag{3.3}$$

where $\psi \sim \psi + 2\pi$. We recognize the metric of an ALE space with an A_1-type singularity. If we define the radial coordinate $\tilde{r} = 2\sqrt{2Rr}$, we obtain a flat space metric with a $\mathbb{Z}_2$ identification.

The two-centers Taub-NUT manifold (3.1), (3.2) possesses a homology 2-cycle given by the fibration of the S^1 parameterized by x^6 over the segment joining the two centers. For coincident centers, in the limit $R \to \infty$, it degenerates to the vanishing 2-cycle of the

$\mathbb{Z}_2$ orbifold located at the origin of the space. The dual of this cycle is an anti-self-dual harmonic 2-form, $G^{(2)}$, defined as

$$G^{(2)} = d\left(V^{-1}\left(dx^6 + 2R\cos\theta d\phi\right)\right) \tag{3.4}$$

and which satisfies the relation

$$G^{(2)} \wedge G^{(2)} = -\Delta(r)\,\Omega_{\text{Taub-NUT}} \tag{3.5}$$

where $\Delta(r) \equiv (\partial V^{-1})^2$ and $\Omega_{\text{Taub}-\text{NUT}}$ is the volume form of Taub-NUT.

To recover the fractional D3-brane solution in the orbifold limit, we will search for a D3-brane metric of the form

$$ds^2_{\text{IIB}} = H^{-1/2} dx^2_{1+3} + H^{1/2}\left(dz d\bar{z} + ds^2_{\text{Taub-NUT}}\right). \tag{3.6}$$

induced by the presence of fluxes of NS-NS and R-R 2-forms through the 2-cycle of the Taub-NUT space.[4] In the orbifold limit, they become twisted NS-NS and R-R scalars. They are expressed in term of the harmonic 2-form:

$$C^{(2)} + iB^{(2)} = \gamma\, G^{(2)} \tag{3.7}$$

and the complex scalar γ is a solution of the two dimensional Laplace equation

$$\Box_{\mathbb{R}^2}\gamma = 0 \tag{3.8}$$

in the space transverse to the D3-branes and to the Taub-NUT manifold. As in the previous section, requiring the preservation of supersymmetries imposes that γ is a holomorphic function. Moreover, we would like to have N fractional D3-branes located at positions z_p. A solution satisfying these constraints is given by

$$\gamma(z) = 4i\pi g_s^B \alpha' \sum_{p=1}^{N} \log\frac{(z+z_p)}{\Lambda} \tag{3.9}$$

but, as discussed in the previous section, it could receive corrections which improve its behaviour close to the D-branes.

Finally, the Einstein equation implies that the harmonic function $H(r, z, \bar{z})$ must verify:

$$\left(\Box_{\mathbb{R}^2} + \Box_{\text{Taub-NUT}}\right) H = -|\partial_z\gamma|^2 \Delta(r)\,. \tag{3.10}$$

A standard way to obtain solutions to this equation is to perform a Fourier transformation over z and $\bar{z}$. For simplicity, we will consider only the case of $z_p = 0$ for which the rotational symmetry in the z-plane is preserved. We call $\rho = \sqrt{z\bar{z}}$. Therefore, the Fourier transform of the harmonic function H is defined by

$$\hat{H}(r,\mu) = \frac{1}{2\pi}\int_0^\infty d\rho\,\rho\, J_0(\mu\rho)\ \left(H(r,\rho) - 1\right). \tag{3.11}$$

[4] An alternative approach would be to search for a "true" D5-brane solution wrapped on the 2-cycle. This has been explored in [6, 7]. There, the NS-NS 2-form is zero while the dilaton has a non-trivial profile. Moreover, the solution does not carry any D3-brane charge.

μ runs from 0 to infinity. For convenience, the constant part of the harmonic function has been subtracted to define its Fourier transform. Then, the field equation (3.10) becomes

$$\left(\partial_r^2 + \frac{2}{r}\partial_r - \mu^2\left(1 + \frac{2R}{r}\right)\right)\hat{H} + \hat{J} = 0\,, \tag{3.12}$$

where $\hat{J}$ is the Fourier transform of $V(\partial_r V^{-1})^2|\partial_z\gamma|^2$.

The general solution of this equation is the sum of the homogeneous equation solutions and of a particular solution. One can see that the homogeneous equation is related to the Whittaker equation; its generic solution is given by

$$\hat{H}_h(r,\mu) = \frac{1}{r}\left(\alpha(R,\mu)W_{-\mu R,-\frac{1}{2}}(2\mu r) + \beta(R,\mu)W_{\mu R,-\frac{1}{2}}(-2\mu r)\right). \tag{3.13}$$

To fix the integration constants α and β, we will impose (hopefully) reasonable D3-brane boundary conditions in the asymptotic regions $r \to \infty$ (R small) and $r \to 0$ (R large).

First, one would like to have an asymptotically flat metric when $r \to \infty$. This implies that $\lim_{r\to\infty}\hat{H}(r,\mu) = 0$. On the other hand, we know the behaviour of the Whittaker function when the norm of its argument is large [53]:

$$W_{a,b}(x) \sim x^a\, e^{-x/2} \quad \text{for} \quad |x| \to \infty\,. \tag{3.14}$$

This means that β must vanish.

Then, we can consider the ALE space limit, namely R large, with $\tilde{r} = 2\sqrt{2Rr}$ fixed, arbitrary, being the radial coordinate of the four-dimensional flat space. At leading order, we would like to recover the first term of the harmonic function of fractional D3-branes [12] at the orbifold conformal field theory point where they carry half of the mass and of the charge of bulk D3-branes,[5]

$$H_h\left(\frac{\tilde{r}^2}{8R},\rho\right) \simeq 1 + \frac{4\pi N g_s^B \alpha'^2}{(\tilde{r}^2+\rho^2)^2}\,, \tag{3.15}$$

whose Fourier transform is expressed in term of the Bessel function K_1 as:

$$\hat{H}_h\left(\frac{\tilde{r}^2}{8R},\mu\right) \simeq \frac{4\pi N g_s^B \alpha'^2}{\tilde{r}}\,\frac{\mu K_1(\mu\tilde{r})}{2}\,.$$

Using the following property of the Whittaker function [53],

$$\lim_{R\to\infty}\left(\Gamma(1+\mu R)\,W_{-\mu R,-\frac{1}{2}}\left(\frac{\mu\tilde{r}^2}{4R}\right)\right) = \mu\tilde{r}K_1(\mu\tilde{r}), \tag{3.16}$$

one finds that $\alpha(R,\mu) \sim 4\pi N g_s^B \alpha'^2\,\Gamma(1+\mu R)/16R$ when R is large. A priori, this formula is valid only in the ALE space limit and does not say anything when R is small. However, one can try to see if we obtain a sensible answer when we extrapolate this expression to small R, where the Taub-NUT space is asymptotically of the form $\mathbb{R}^3 \times S^1_{4\pi R}$.

[5]This expression is the same as for N standard D3-branes in a flat space. However, one must remember that the $\mathbb{Z}_2$ identification $\psi \sim \psi + 2\pi$ divides by two the volume of the surrounding 5-sphere used to compute the (untwisted) D3-brane charge and, therefore, gives only an $N/2$ D3-brane charge.

Therefore, let us consider as the full solution of the homogeneous equation the following expression:

$$\hat{H}_h(r,\mu) = \frac{4\pi N g_s^B \alpha'^2\,\Gamma(1+\mu R)}{16Rr} W_{-\mu R,-\frac{1}{2}}(2\mu r) \tag{3.17}$$

and study its small R behaviour. If we expand this formula in the asymptotic region R small, $r \to \infty$, we obtain:

$$\hat{H}_h(r,\mu) \sim \frac{4\pi N g_s^B \alpha'^2\,\Gamma(1+\mu R)}{16Rr}\, e^{-\mu r}\,(1+\mu R\log(2\mu r)+\cdots) \tag{3.18}$$

Its leading order in an expansion in series of R can be Fourier transform back to:

$$H_h(r,\rho) \sim 1 + \frac{4\pi N g_s^B \alpha'^2}{16R(r^2+\rho^2)^{3/2}} \tag{3.19}$$

In this expression, we recognize the smeared harmonic function of $N/2$ D3-branes delocalized along the circle $S^1_{4\pi R}$. We see that the division by two of the D3-brane charge imposed in the ALE space limit by choosing a fractional D3-brane boundary condition has propagated to the other limit where the Taub-NUT space degenerates to $\mathbb{R}^3 \times S^1_{4\pi R}$ with R small. Actually, when R is sent to zero, the type-IIB description is not the correct one anymore and one must go to its type IIA dual. In the next section, we will perform explicitly this T-duality and we will show that the division by two confirms the interpretation of the T-dual configuration as D4-branes stretched between NS5-branes, carrying half of the charge of standard D4-branes.

However, before doing this, we must mention that the full solution could be obtained by adding to (3.17) a particular solution of the equation (3.12). To find such a solution one can use the method of [18]: one constructs first the Green function $G(\mu|r,r')$ with the appropriate boundary conditions using the solutions of the homogeneous equation. Then, the particular solution will be given by

$$\hat{H}_{n-h}(r,\mu) = -\int_0^\infty G(\mu|r,r')\,\hat{J}(r',\mu)\,(r')^2 dr'\,. \tag{3.20}$$

3.2 Uplift to eleven dimensions and comments

Using transformation rules for NS-NS and R-R fields given in [54], we can perform a T-duality along the isometric direction x^6 to map the above solution to a type-IIA one. Discussions in the literature have argued that the dual of the two-centers Taub-NUT space is given by two NS5-branes whose separation in the x^6 direction is related to the twisted scalar b and that the fractional D3-branes become D4-branes stretched between the two NS5-branes. The T-dual coordinate $\tilde{x}^6$ has periodicity $\pi\alpha'/R$ and the type-IIA gauge coupling constant is $g_s^A = g_s^B\sqrt{\alpha'}/2R$. The D4-branes/NS5-branes metric, dilaton, NS-NS and R-R field strength read:

$$
\begin{aligned}
ds^2_{\mathrm{IIA}} &= H^{-1/2}dx^2_{1+3} + H^{1/2}\left(dzd\bar{z} + V\delta_{ij}dx^i dx^j\right) + H^{-1/2}V\left(d\tilde{x}^6 - b\,dV^{-1}\right)^2 ,\\
e^{\phi_A} &= VH^{-1/2} , \qquad H^{(3)} = \frac{\alpha'}{\tilde{R}}\sin\theta\, d\left(\tilde{x}^6 - b\,V^{-1}\right)\wedge d\theta\wedge d\phi ,\\
F^{(2)} &= -dc\wedge dV^{-1} ,\\
F^{(4)} &= \frac{\alpha'}{\tilde{R}}\sin\theta\, V^{-1}\left(\left(d\tilde{x}^6 - b\,dV^{-1}\right)\wedge dc + 2c\,db\wedge dV^{-1}\right)\wedge d\theta\wedge d\phi ,\\
F^{(6)} &= dH^{-1}\wedge dx^0\wedge\cdots\wedge dx^3\wedge\left(d\tilde{x}^6 - b\,dV^{-1}\right) ,\\
\gamma(z) &= (c+ib)(z) = 4\pi i N g_s^A\frac{(\alpha')^{3/2}}{\tilde{R}}\log\frac{z}{\Lambda} ,
\end{aligned}
\tag{3.21}
$$

where we have defined the dual radius $\tilde{R} = \alpha'/2R$. As always after T-duality, the p-forms appear at the same time as the dual $10-p$-forms. However, one can transform the 4-form $F^{(4)}$ to its Hodge dual and combine the result with the $F^{(6)}$ of (3.21). More surprising is the existence of a non trivial R-R 2-form $F^{(2)}$. It comes from the dualization of the 2-form potential $C^{(2)}_{x^6 r}$ which was required to obtain a solution which preserves supersymmetry. Dualizing this 2-form field strength to an 8-form gives:

$$F^{(8)} = (HV)^{-1}dx^0\wedge\cdots\wedge dx^3\wedge db\wedge H^{(3)} . \tag{3.22}$$

Therefore, we see that the R-R 7-form potential vanishes and the 8-form is induced by the non-trivial NS-NS 3-form field strength $H^{(3)}$ and R-R 5-form potential $C^{(5)}_{x^0x^1x^2x^3\rho}$.

Since the type-IIA solution has been obtained using T-duality in the x^6 direction, the two NS5-branes are delocalized along this coordinate and their harmonic function is smeared:

$$V = 1 + \frac{\alpha'}{\tilde{R}r} .$$

In particular, momentum modes along $\tilde{x}^6$ have been neglected [33]. Moreover, this NS5-brane harmonic function does not include the back-reaction due to the stretched D4-branes.

Now, we can express the leading order (3.19) of the harmonic function in term of type-IIA variables:

$$H_h(r,\rho) \sim 1 + \frac{\pi N g_s^A(\alpha')^{3/2}}{2(r^2+\rho^2)^{3/2}} . \tag{3.23}$$

This harmonic function corresponds to N D4-branes with half of the mass and charge of standard D4-branes as one can expect for D4-branes stretched between NS5-branes located at antipodal points of the circle.[6] This result is an argument in favor of the conjecture that the expression (3.17) is correct not only for large R but also for any radius.

Then, we can uplift this solution to eleven dimensions. The type-IIA and eleven dimensional metrics are related by

$$ds^2_{11d} = e^{-2\phi_A/3}ds^2_{\mathrm{IIA}} + e^{4\phi_A/3}\left(dx^{10} + C^{(1)}_\mu dx^\mu\right)^2 \tag{3.24}$$

[6]The approximation (3.23) corresponds to this symmetric configuration since the bending of the NS5-branes due to the stretched D4-branes appears only when one takes into account the non-homogeneous solution.

which leads to the result

$$ds^2_{11d} = (HV)^{-1/3}dx^2_{1+3} + (HV)^{2/3}\left(dr^2 + r^2 d\Omega_2^2\right) + H^{2/3}V^{-1/3}dz d\bar{z} + \\ + H^{-1/3}V^{2/3}\left(\left(d\tilde{x}^6 - b\,dV^{-1}\right)^2 + \left(dx^{10} + c\,dV^{-1}\right)^2\right)\,. \tag{3.25}$$

To relate this solution to the wrapped M5-branes configurations considered in [29], we rewrite the metric as a function of the new complex coordinate $z^2 = s$ defined as:

$$s \equiv \tilde{x}^6 + ix^{10} + i\gamma(z)\,V^{-1}(r)\,. \tag{3.26}$$

Renaming z into z^1, the metric takes the form of the Fayyazuddin/Smith ansatz [55, 56]:

$$ds^2_{11d} = g^{-1/3}dx^2_{1+3} + g^{2/3}\left(dr^2 + r^2 d\Omega_2^2\right) + g^{-1/3}g_{m\bar{n}}dz^m dz^{\bar{n}} \tag{3.27}$$

where $g = H\,V$ is the determinant of the Kähler metric

$$g_{z\bar{z}} = H + V^{-1}|\partial\gamma|^2\,, \qquad g_{s\bar{s}} = V\,, \qquad g_{z\bar{s}} = -i\partial\gamma\,, \qquad g_{s\bar{z}} = i\bar{\partial}\bar{\gamma}\,. \tag{3.28}$$

It is easy to check that this metric verifies the equations:

$$\partial_m\partial_{\bar{n}}g + \Box_3 g_{m\bar{n}} = 0\,. \tag{3.29}$$

In particular, the equation (3.29) for $(m,\bar{n}) = (1,\bar{1})$ is just a rewriting of (3.10).

To summarize, starting with the type-IIB supergravity solution which describes fractional D3-branes on a two-centers Taub-NUT space, we have obtained a partially smeared supergravity solution representing M5-branes wrapped on a two dimensional surface parameterized by the coordinates s and z. Since the gauge theory living on the world-volume of the D3-branes is a pure $\mathcal{N} = 2$ SU(N) SYM theory, the eleven dimensional solution should also correspond to this theory and one expects that the the Riemann surface is given by the Seiberg-Witten curve of this gauge theory.

This problem has been studied directly in eleven dimensions in [29], where a completely localized solution has been proposed. Let us remind in few words their result. The authors of [29] considered a configuration of M5-branes wrapped on the Riemann surface defined by

$$f(t,z) = t + 2B(z) + \frac{1}{t} = 0 \tag{3.30}$$

which is the Seiberg-Witten curve of the pure $\mathcal{N} = 2$ SU(N) SYM theory when $B(z) = \prod_{p=1}^{N}(z - z_p)$ [57]. In this case, only the eleventh coordinate is compactified $x^{10} \sim x^{10} + 2\pi R'$ and $t = e^{s/R'}$. They consider a "decoupling" limit, where field theory quantities are held fixed while the eleven dimensional Planck scale is sent to zero. The new variables Z, S and u defined by

$$Z = \frac{z}{\alpha'} = \frac{zR'}{l_P^3}\,, \qquad S = \frac{s}{R'}\,, \qquad u^2 = \frac{r}{l_P^3} \tag{3.31}$$

are fixed under the scaling. In term of these new variables, the Kähler potential which defines the metric appears as the sum of two terms $K = K^{(1)}(u, F(Z,S), \bar{F}(\bar{Z},\bar{S})) + |G(Z,S)|^2$:

$$K^{(1)}(u, F(Z,S), \bar{F}(\bar{Z},\bar{S})) = \frac{\pi N}{u^2} \ln \frac{\sqrt{u^4 + |F|^4} + u^2}{\sqrt{u^4 + |F|^4} - u^2}, \qquad F = f^{1/N},$$
$$G(Z,S) = -\frac{Z N f^{1-\frac{1}{N}}}{4} \int_0^1 \frac{dt}{\sqrt{(f/2 - B(tZ))^2 - 1}}. \tag{3.32}$$

Using this Kähler potential, one can calculate the determinant of the metric:

$$g = \frac{\pi N}{2\left(u^4 + |F|^4\right)^{3/2}}. \tag{3.33}$$

This metric is singular at the location of the M5-brane, namely on the Seiberg-Witten curve at $r = 0$. However, it does not have the repulsive singularities displayed by other $\mathcal{N} = 2$ solutions. It would be interesting to probe this background to see if one can reproduce the running of the coupling constant on the world-volume of the probe. To do this, one should consider the action of M5-brane wrapped on the Riemann surface defined by (3.30).

Unfortunately, we see that the two solutions are obtained in two very different approximations. Indeed, in the D3-brane case, s parameterizes a 2-torus while the Seiberg-Witten curve (3.30) is valid only when $\tilde{x}^6$ is a non-compact coordinate. Moreover, the M5-brane solution considered here is smeared over this 2-torus, whereas the solution (3.33) of [29] was completely localized. Since the functions V and γ are smeared and depend explicitly on the radius of compactification, the decompactification limit is meaningless. The last difference is that our metric is asymptotically flat while the M5-brane of [29] is only a "near-horizon" solution. One can take also a near horizon limit in (3.25) but the other approximations and hypothesis make the comparison impossible. Obviously, it would be very interesting to find a solution which goes beyond the limits of these two solutions, namely which is fully localized on the 2-torus and is not just valid in a decoupling region but the task seems to be difficult. A better understanding of the properties of the solution of [29], in particular in the optic of the gauge/gravity correspondence, is also desirable.

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